Robust Uncertainty Principle

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Outline

Fourier Analysis on Cyclic Groups

Fourier Analysis on Finite Abelian Groups

Properties of Fourier Transforms

Dual Group of Fourier Transform

Uncertainty Principles

UCP on p-norms

Donoho-Strak UCP

Robust Version of UCP.

Cyclic Group $\mathbb{Z}_n$

$\mathbb{Z}/n\mathbb{Z} := \{0, 1, ..., n-1\}$ with $+ \ mod(n)$

Definition: Character

Character of group G is Continuous homomorphism from G to circle group $\mathbb{T} := \{e^{i\theta} : 0 \leq \theta < 2\pi\}$

Cyclic Group $\mathbb{Z}_n$

Definition

Character of group G is Continuous homomorphism $\chi : G \to \mathbb{T} := \{e^{i\theta} : 0 \le \theta < 2\pi\}$

- Q: What are the characters of $\mathbb{Z}_n$?
  $\mathbb{Z}_n$ is cyclic $\implies f(k) = f(1)^k$
  and $f(1)^n = f(0) = 1 \implies f(1) \in \{e^{2\pi i \frac{k}{n}} : 0 \le k \le n-1\}$
- $\therefore$ for each $k \in \{0, 1, \ldots, n-1\}$,

$$\chi_k(j) = e^{2\pi i \frac{jk}{n}} \quad \text{for } j \in \mathbb{Z}_n.$$

- These are the all homomorphisms of the $\mathbb{Z}_n$ into $\mathbb{T}$.

Space $L^2(G)$

Definition: $L^2(G)$

Let G be a finite group($|G| = N$), we define $L^2(G) := \{f : G \to \mathbb{C}\}(\cong \mathbb{C}^N)$

- $L^2(G)$ is a N-dim complex vector space with

$$\langle f, g \rangle = \sum_{x \in G} f(x)\overline{g(x)} \quad \leftsquigarrow \text{ inner product} \tag{1}$$

$$\|f\|^2_{L^2(G)} = \sum_{x \in G} |f(x)|^2 \quad \leftsquigarrow \text{ Norm on the space} \tag{2}$$

Space $L^2(\mathbb{Z}_n)$

- Characters on $L^2(\mathbb{Z}_n)$:

$$\chi_k(j) = e^{2\pi i \frac{jk}{n}} \quad \text{where } k \in \{0, ..., n-1\},\ j \in \mathbb{Z}_n.$$

- character $\chi_0, \chi_1, ...\chi_{n-1}$ are orthogonal in $L^2(\mathbb{Z}_n)$

$$\langle \chi_k\ ,\ \chi_l \rangle_{L^2(\mathbb{Z}_n)} = \sum_{j=0}^{n-1} \chi_k(j)\overline{\chi_l(j)} = \sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n}(k-l)} = \begin{cases} n & if k = l \\ 0 & if k \neq l \end{cases}$$

- $|\{\chi_k\}| = n$

$\therefore$ $\{\chi_k : 0 \leq k \leq n-1\}$ makes basis of $L^2(\mathbb{Z}_n)$.

$$\left\{ \frac{1}{\sqrt{n}} \chi_k : 0 \leq k \leq n-1 \right\} \text{ is an orthonormal basis of } L^2(\mathbb{Z}_n)$$

.

Fourier transforms on $L^2(Z_n)$

- Let $f \in L^2(\mathbb{Z}_n)$, then we can expand f in terms of basis elements χ_k.

$$f = \sum_{k=0}^{n-1} \hat{f}(k) \frac{\chi_k}{\sqrt{n}} \tag{3}$$

$$\text{Where } \hat{f}(k) = \langle f\ ,\ \frac{\chi_k}{\sqrt{n}} \rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f(j) e^{-2\pi i \frac{jk}{n}} \tag{4}$$

Definition: L

et $f \in L^2(\mathbb{Z}_n)$ and let $\hat{f} : \{0, 1, ...n-1\} \to \mathbb{C}$ be function as above, then $\hat{f}$ is called(discrete) Fourier transform of f.

Note that $f \to \hat{f}$ is a map from $L^2(\{0, 1, ...n-1\})$ to itself.

Properties

Theorem

Let $f \in L^2(\mathbb{Z}_n)$ and $\hat{f}$ be its fourier transform then,

$$\|f\|^2_{L^2\{0,1,\dots n-\}} = \sum_{k=0}^{n-1} |f(k)|^2 = \sum_{k=0}^{n-1} |\hat{f}(k)|^2 = \|\hat{f}\|^2_{L^2\{0,1,\dots n-\}} \quad \textbf{Plancherel relation} \tag{5}$$

more generally, $\langle f, g \rangle_{L^2(\mathbb{Z}_n)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{Z}_n)}$ (6)

$\therefore$ map $f \to \hat{f}$ is a unitary transformation.

Properties

Definition: Check$^\vee$

For $g \in L^2(\{0, 1, \ldots n-1\})$, define

$$\check{g}(k) = 1/\sqrt{n} \sum_{j=0}^{n-1} g(j) e^{2\pi i \frac{jk}{n}}.$$

Theorem

Let $f \in L^2(\mathbb{Z}_n)$, Then $\check{\hat{f}} = f$. And $\hat{\check{f}}(j) = f(-j)$.

Examples:

1. Let $\delta_{x_0}(x) = 1$ if $x = 0$ otherwise 0, then

$$\hat{f}(k) = \frac{1}{\sqrt{n}}\langle f, \chi_k \rangle = \frac{\chi_k(0)}{\sqrt{n}} = \frac{1}{\sqrt{n}}$$

2. Let $f = 1$, then

$$\hat{f}(k) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f(j)\overline{\chi_k(j)} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{2\pi i \frac{jk}{n}} = \begin{cases} \sqrt{n} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

3. Let $N = pq$, $p, q \in \mathbb{N}$ Let $f(j) = 1$ if j is multiple of p, otherwise 0. Then

$$\hat{f}(k) = \frac{1}{\sqrt{n}} \sum_{l=0}^{q-1} \overline{\chi_k(l * p)} = \frac{1}{\sqrt{n}} \sum_{l=0}^{q-1} e^{-2\pi i \frac{klp}{pq}} = \begin{cases} \frac{q}{\sqrt{n}} & \text{if } k = 0 \mod q \\ 0 & \text{if } k \neq 0 \mod q \end{cases}$$

Generalization

Q: Can we extend this formulation for general finite group?

- Let G be a finite group ($|G| = N$) and $\chi : G \to \mathbb{T}$ is a character of G.
  $\implies \chi(xyx^{-1}) = \chi(x)\chi(y)\chi(x)^{-1} = \chi(y)$
  $\implies \chi$ is constant on a conjugacy classes.
  Since, $\dim(span\{x\}) \leq$ number of conjugacy classes
  $\implies$ For character to form basis of $L^2(G)$, **G must be abelian.**

Abelian Groups

Q: Can all finite abelian groups have full basis of character?

Theorem

Let G_1, G_2 be two finite groups, and suppose $G = G_1 \times G_2 = \{(x, y | x \in G_1, y \in G_2)\}$ and let $\chi = \chi_1 \otimes \chi_2$ i.e.

$$\chi : G \to \mathbb{T} \quad \text{such that } \chi(x, y) = \chi_1(x)\chi_2(y),$$

then χ is a character of G.

Note that for each distinct pair of (i, j), $\chi_i \otimes \chi_j$ gives distinct character on G.

Theorem

Let group G_1 has $|G_1| = N_1$ character and group G_2 has $|G_2| = N_2$ characters, then $G = G_1 \times G_2$ has $|G| = N_1 N_2$ characters. i.e. G has a full basis of character.

Abelian Group

Theorem: Structure Theorem

Let G be any finite abelian group $|G| = N$, then it is isomorphic to direct sum form;

$$G \cong \mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times \ldots \times \mathbb{Z}_{{p_k}^{n_k}},$$

where p_i is prime and $n_i \geq 1$ s.t $N = {p_1}^{n_1} \ldots {p_k}^{n_k}$.

Theorem

If G is finite abelian group then it has $|G|$ linearly independent characters. In fact they are orthogonal in $L^2(G)$

- If G is abelian then $L^2(G)$ has full basis of character of G.

Fourier Transform on G

G - finite abelian group with $|G| = N$, $\hat{G}$ - set of characters of G,
$\{\frac{\chi}{\sqrt{N}} : \chi \in \hat{G}\}$ is orthonormal basis for $L^2(G)$.

Suppose Suppose $f \in L^2(G)$, then we can write

$$f = \sum_{\chi \in \hat{G}} \langle f, \frac{\chi}{\sqrt{N}} \rangle \frac{\chi}{\sqrt{N}} \quad = \frac{1}{\sqrt{N}} \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$$

Definition: Fourier Transform

Let G be an abelian group, for $f : G \to \mathbb{C}$ we define $\hat{f} : \hat{G} \to \mathbb{C}$ such that

$$\hat{f}(\chi) = \frac{1}{\sqrt{N}} \sum_{x \in G} f(x)\overline{\chi(x)},$$

then $\hat{f}$ is called fourier transform of f and map $f \to \hat{f}$ is fourier transform on G.

Properties

- **Plancerel relation:** $\mathfrak{F} : L^2(G) \to L^2(\hat{G})$ is unitary. i.e.

$$\langle f, g \rangle_{L^2(G)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\hat{G})} \tag{7}$$

$$\|f\|^2_{L^2(G)} = \|\hat{f}\|^2_{L^2(\hat{G})} \tag{8}$$

- **Translation operator:** For any $x \in G$, let $T_x : L^2(G) \to L^2(G)$ s.t. $(T_x f)(y) = f(x + y)$.
  $T_x \chi = \chi(x)\chi$. so χ is an eigenvector of T_x with eigenvalue $\chi(x)$.
- **Fourier inversion:** For $f \in L^2(G)$, $\check{f}(\chi) = \frac{1}{\sqrt{N}} \langle f, \overline{\chi} \rangle = \frac{1}{\sqrt{N}} \sum_{x \in G} f(x)\chi(x)$
  Then $\overset{\times}{\hat{f}} = f$.

Dual Group

Q: What is relation between G and $\hat{G}$?

- $\hat{G}$ is finite abelian group. with $(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$, for any $\chi_1, \chi_2 \in \hat{G}$.
  - $\chi_1\chi_2$ is a homomorphism from G to $\mathbb{T}$
    $\because \chi_1\chi_2(x+y) = \chi_1(x)\chi_1(y)\chi_2(x)\chi_2(Y) = \chi_1\chi_2(x)\chi_1\chi_2(y)$
  - $\chi^{-1} \in \hat{G}$ as $\chi^{-1}(x+y) = (\chi(x)\chi(y))^{-1} = \chi(x)^{-1}\chi(y)^{-1}$
  - identity of $\hat{G}$ is the trivial character
- $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times ... \times \mathbb{Z}_{n_p}$, $\hat{G} = \hat{\mathbb{Z}}_{n_1} \times \hat{\mathbb{Z}}_{n_2} \times ... \times \hat{\mathbb{Z}}_{n_p}$. So there is isomorphism from $(k_1, k_2, ...K_p) \to \chi_{k_1} \otimes \chi_{k_2} \otimes ... \otimes \chi_{k_p}$. So $\hat{G} \cong G$
- But there is no canonical or natural isomorphisms between G and $\hat{G}$.

What about relation between G and $\hat{\hat{G}}$?

$G \cong \hat{\hat{G}}$

$\hat{\hat{G}}$ is set of characters of $\hat{G}$. For $x \in G$ we define the evaluation map

$$ev_x : \hat{G} \longrightarrow \mathbb{T}$$
$$ev_x(\chi) = \chi(x)$$

Then $(ev_x(\chi_1\chi_2)) = ev_x(\chi_1)ev_x(\chi_2)$. Hence $ev_x \in \hat{\hat{G}}$.

- Map $G \to \hat{\hat{G}}$ which takes $x \to ev_x$ is homomorphism. And injective as $\chi(0) = 1 \forall \chi$.
- $L^2(\hat{\hat{G}}) = L^2(G)$
- $\hat{\hat{f}}(x) = f(-x)$

Uncertainty Principles

What is the uncertainty principles?

Let's recall a L^p norm;

$$\|f\|_p = \left(\sum_{x \in G} |f(x)|^p \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty, \quad \|f\|_\infty = \max_{x \in G} |f(x)|$$

Also recall that; If G is finite abelian group $|G| = N$ and $\hat{G}$ is its dual and $f \in L^2(G)$, $\hat{f} \in L^2(\hat{G})$. Then

$$f(x) = \frac{1}{\sqrt{N}} \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(x) \tag{9}$$

$$\hat{f}(x) = \frac{1}{\sqrt{N}} \sum_{x \in G} f(x)\overline{\chi(x)} \tag{10}$$

Note that $|\chi(x)| = 1$ for all $x \in G$ and $\chi \in \hat{G}$

Uncertainty Principle

Taking absolute values of (9) and (10); we get $|f(x)| \leq \frac{1}{\sqrt{N}} \sum_{\chi \epsilon \hat{G}} |\hat{f}(\chi)|$

$$\implies \quad ||f||_\infty \leq \frac{1}{\sqrt{N}} ||\hat{f}||_1 \tag{11}$$

$$\text{similarly,} \quad ||\hat{f}||_\infty \leq \frac{1}{\sqrt{N}} ||f||_1 \tag{12}$$

By multiplying (11) and (12) and rearranging terms we get; the First Uncertainty inequality:

$$\frac{||f||_1}{||f||_\infty} \times \frac{||\hat{f}||_1}{||\hat{f}||_\infty} \geq N \tag{13}$$

Uncertainty Principle

$$\frac{||f||_1}{||f||_\infty} \times \frac{||\hat{f}||_1}{||\hat{f}||_\infty} \geq N$$

Q:What this exactly means?

- It says that f and $\hat{f}$ cannot be both simultaneously "concentrated".
- We can say δ_{x_0} is concentrated at one point and on other hand the constant function $\mathbb{1}$ is not concentrated.

Q How the ratio of norm $||f||_p/||f||_q$ can be taken as a measure of the spread?

- We know $||f||_1 \leq ||f||_\infty$

  $$\frac{||f||_1}{||f||_\infty} = \begin{cases} 1 & \text{if } f = f = \delta_{x_0} \\ N & \text{if } f = \mathbb{1} \end{cases}$$

- So the ration of L^1 and L^∞ norm shows sort of concentration in this sense.
- in general we 'conclude' that p to q norm ratio measures the spread of function in some sense.

Donoho-Stark UCP

Definition: Support of a function

Let f be a function on G then the $S_f \subseteq G$ defined as;

$$S_f := \{x \in G : f(x) \neq 0\}$$

is support of f. and $|S_f|$ denotes the cardinality of support.

Let f be a function and $\hat{f}$ be its fourier transform with support S_f and $S_{\hat{f}}$, then

- $\|f\|_1 = \sum_{x \in S_f} |f(x)|$ as $f = 0$ outside S_f.
- for any $x \in G$, $|f(x)| \leq \|f\|_\infty$

Using this in (12) gives us the inequality,

$$\|\hat{f}\|_\infty \leq \frac{1}{\sqrt{N}}\|f\|_1 \leq \frac{1}{\sqrt{N}}\|f\|_\infty |S_f| \tag{14}$$

Donoho-Strak UCP

$$\|\hat{f}\|_\infty \leq \frac{1}{\sqrt{N}}\|f\|_\infty |S_f| \quad \text{and} \quad \|f\|_\infty \leq \frac{1}{\sqrt{N}}\|\hat{f}\|_\infty |S_{\hat{f}}|. \tag{15}$$

By multiplying both we get

Theorem: Donoho-Stark Inequality

Let G be an abelian group $|G| = N$, let $f : G \to \mathbb{C}$ be a function on G and $\hat{f} : \hat{G} \to \mathbb{C}$ be its fourier transform. then,

$$|S_f| \times |S_{\hat{f}}| \geq N$$

Donoho-Strak UCP

Theorem: Donoho-Stark Inequality

Let G be an abelian group $|G| = N$, let $f : G \to \mathbb{C}$ be a function on G and $\hat{f} : \hat{G} \to \mathbb{C}$ be its fourier transform. then,

$$|S_f| \times |S_{\hat{f}}| \geq N$$

Equality can be achieved, if $N = pq$ and $G = \mathbb{Z}_n$ and

$$f(j) = \begin{cases} 1 & \text{if } j = 0 \mod p \\ 0 & \text{otherwise.} \end{cases} \qquad \hat{f}(k) = \begin{cases} q/\sqrt{N} & \text{if } k = 0 \mod q \\ 0 & \text{otherwise} \end{cases}$$

- here $|S_{\hat{f}}| = p, |S_f| = q$ Therefore we get $|S_f| \times |S_{\hat{f}}| = N$.

Problem in Application

- **The conclusions of the mathematical theory depend on the precise validity of the assumptions.**

- If we have a sound signal for a duration, we can capture it as a function on an interval. or image can be captured as a function on a square.
- In the real world, no function is anywhere ever exactly zero but only approximately.

- Theorem like Donoho-stark will be vacuous because every function have full support. i.e.

$$|S_f| \times |S_{\hat{f}}| = N^2$$

Robust Theorem

A Robustness/stability of theorem determines the extent to which the conclusions of a given theorem are affected if the assumptions of the theorem are not exactly but only approximately satisfied

- Q: Would the Donoho-Stark UCP hold in the stable version?
- we must check whether can we have two sets both of which are small such that f and $\hat{f}$ are approximately supported on them, respectively.

- The meaning of the word 'approximately' has to be defined exactly.

The Approximate Support

Definition: (p, δ)–support

Let G be a finite abelian group, $|G| = N$ and let $f : G \to \mathbb{C}$ be function in $L^2(G)$. Let fix $p \in (0, \infty)$ and $\delta \in (0, \infty)$. A set $A \subseteq G$ is said to be an approximate (p, δ)–support for f, if

$$||f \mathbb{1}_A||_p \geq (1 - \delta)||f||_p$$

.

- Support of a function is an unique set, but (p, δ)–support is not unique.
- whole set G is always a (p, δ)–support.

Robust Donoho-Stark UCP

Theorem: Donoho-Stark at $(1, \varepsilon)$ approximation

Let G be a finite abelian group and let $f : G \to \mathbb{C}$ be a function on G and $\hat{f} : \hat{G} \to \mathbb{C}$ be its fourier transform. Suppose $A \subseteq G$ is a $(1, \varepsilon)$ approximate support for f and $B \subseteq \hat{G}$ is a $(1, \delta)$ approximate support for $\hat{f}$. Then

$$|A| \times |B| \geq N(1 - \varepsilon)(1 - \delta) \tag{16}$$

proof.

- $\|f\mathbb{1}_A\|_1 = \sum_{x \in A} |f(x)| \leq \|f\|_\infty |A|$ and $\|f\|_1 \leq \frac{\|f\mathbb{1}_A\|_1}{(1-\varepsilon)}$ (by def of supp.)
- by inequality (16); $\|\hat{f}\|_\infty \leq \frac{1}{\sqrt{N}(1-\varepsilon)} \|f\|_\infty |A|$ and $\|f\|_\infty \leq \frac{1}{\sqrt{N}(1-\delta)} \|\hat{f}\|_\infty |B|$
- By multiplying both we get the result.

Robust Donoho-Stark UCP

Theorem: Donoho-Stark at $(2, \delta)$ approximation

Let G be a finite abelian group of order N. Suppose $f : G \to \mathbb{C}$ is not identically zero. Suppose $A \subseteq G$ and $B \subseteq \hat{G}$ are $(2, \varepsilon)$ and $(2, \delta)$ support for f and $\hat{f}$ respectively. Then

$$|A| \times |B| \geq N(1 - \varepsilon - \delta)^2 \tag{17}$$

sketch of the proof.

- Let $T : V \to V$ be linear operator on finite dim vector space V with an operator norm $\|\cdot\|$ and $\|T\|_F = \sqrt{tr(T^* T)}$ be Frobenious norm. Then we have following inequality:

$$\|T\| \leq \|T\|_F$$

Robust Donoho-Stark UCP

sketch of the proof.

- Let's introduce projection operators on $L^2(G) \to L^2(G)$:

$$P_A := \text{ restriction of } f \text{ to A } \subseteq G : P_A f = f \mathbb{1}_A$$

$$\hat{P_B} := \text{restriction of } \hat{f} \text{ to } B \subseteq \hat{G} : \hat{P_B} f = (\hat{f} \mathbb{1}_B)\check{} \text{ or } (\hat{P_B} f)\hat{} = \hat{f} \mathbb{1}_B$$

$$\left\| \hat{P}_B P_A f \right\|_2 \geqslant (1 - \varepsilon - \delta) \|f\|_2 \quad ...(C1)$$

$$\sqrt{\frac{|A| \times |B|}{N}} = \left\| \hat{P}_B P_A \right\|_F \geqslant \left\| \hat{P}_B P_A \right\| \quad ...(C2)$$

$$\implies \sqrt{\tfrac{|A| \times |B|}{N}} \geqslant (1 - \varepsilon - \delta)$$

Robust Donoho-Stark UCP

sketch of the proof.

- Form approximate support condition:

$$\|P_A f\|_2 \geq (1-\delta)\|f\|_2 \quad \text{and} \quad \|\hat{P}_B f\|_2 \geq (1-\varepsilon)\|f\|_2$$

- Consider;

$$\begin{aligned}
\left\|\hat{P}_B P_A f\right\|_2 &\geqslant \left\|\hat{P}_B f\right\|_2 - \left\|\hat{P}_B (I-P_A) f\right\|_2 \quad \left(\because P_A f = f-(I-P_A)\right) \\
&\geqslant (1-\epsilon)\|f\|_2 - \|(I-P_A) f\|_2 \quad \left(\because \|\hat{P}_B g\|_2 \leq \|g\|_2 \text{ as } \hat{P}_B \text{ is a projection}\right) \\
&\geqslant (1-\varepsilon-\delta)\|f\|_2 \quad \left(\because \|(I-P_A) f\|_2 \leqslant \delta \|f\|_2\right)
\end{aligned}$$

- Now for the second claim; $\left\|\hat{P}_B P_A\right\| = Tr(\hat{P}_B P_A)$ (To compute trace we use basis $\delta_k, k \in G$.) If $k \in A$, $\hat{P}_B P_A \delta_k = \hat{P}_B \delta_k = (\mathbb{1}_B \hat{\delta}_k)$, otherwise 0.
$\therefore (k,k)$ entry of $\hat{P}_B P_A = \frac{1}{N}|B|$. summing over $k \in A$; $\left\|\hat{P}_B P_A\right\|_F^2 = \frac{1}{N}|A| \times |B|$.

Robust Donoho-Stark UCP

Theorem

Let G be a finite abelian grop of order N. suppose $f : G \to \mathbb{C}$ is not identically zero. Suppose $A \subseteq G$ and $B \subseteq \hat{G}$ are $(1, \varepsilon)$ and $(2, \delta)$ support for f and $\hat{f}$ respectively. Then

$$|A| \times |B| \geq N(1-\varepsilon)^2(1-\delta)^2 \tag{18}$$

Thank You!