Plus Topology and it's properties

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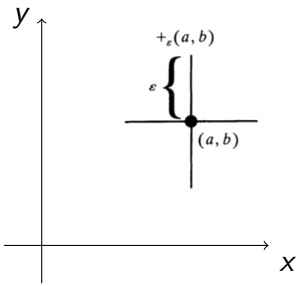
Outline

- ϵ Plus
  - Plus Topology on $\mathbb{R}^2$
  - Structure of Plus open sets
- Properties of Plus open sets
- Separately Continuous Functions
- Separation axioms
- Non-regularity of $\mathbb{R} \otimes \mathbb{R}$
  - Neighbourhood base of $\mathbb{Q} \otimes \mathbb{Q}$

ε - Plus

For $\varepsilon > 0$ and $(a, b) \in \mathbb{R}^2$, P_ε centered at (a, b) is the set:

$$P_\varepsilon(a, b) = \{(x, b) \in \mathbb{R}^2 \mid |x - a| < \varepsilon\} \cup \{(a, y) \in \mathbb{R}^2 \mid |y - b| < \varepsilon\}$$



- More generally, if X and Y are topological spaces, $(p, q) \in X \times Y$, and U is neighbourhood of p in X and V is neighbourhood of q in Y, then we define $(U, V)-plus$ at (p, q) by

$$P_{U,V}(p, q) = \{(x, q) : x \in U\} \cup \{(p, y) : y \in V\}$$

Plus Topology on $\mathbb{R}^2$

Definition

$E \subseteq \mathbb{R}^2$ is **plus-open** if for every $(a, b) \in E$ there is some $\varepsilon > 0$ s.t. $P_\varepsilon(a, b) \subseteq E$.

- The Plus topology, $\mathcal{T}_+$ is the collection of all plus-open sets.
- $\mathbb{R} \otimes \mathbb{R}$ denotes $\mathbb{R} \times \mathbb{R}$ with the topology $\mathcal{T}_+$.

Definition

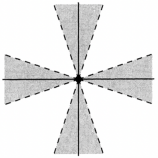
A function f with domain $\mathbb{R}^2$ is called **Separately Continuous** if for every $b \in \mathbb{R}$, $f(x, b)$ is continuous function of x, and for every $a \in \mathbb{R}$, $f(a, y)$ is continuous function of y.

Theorem

Let Y be topological space, and $f : \mathbb{R}^2 \mapsto Y$, then f is *separately continuous* iff. it is continuous w.r.t. plus topology on $\mathbb{R}^2$. furthermore plus topology is the only topology for which this is true.

Structure of Plus open sets

- Example 1: Maltese cross
  $A = \{0,0\} \cup \{(x,y) \mid |y| > 3|x|\} \cup \{(x,y) \mid |y| < \frac{|x|}{3}\}$
  A is plus-open but not an Euclidean open set.
  - Plus-topology is finer than Euclidean topology on $\mathbb{R}^2$



- Example 2:
  - Let p be a point on some circle in plane.
  - Let remove the circle from plane except the point p.
  - The remaining set is Plus open but point p is not in the euclidean interior.
- Example 3:
  - Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$ and $\frac{\alpha}{\beta} \notin \mathbb{Q}$.
  - Let f be a rotation function given by

$$f(x,y) = (\alpha x + \beta y, -\beta x + \alpha y).$$

    This f rotates point (x,y) by angle θ clockwise, where $\sin\theta = \beta, \cos\theta = \alpha$.
  - Every horizontal and vertical line intersect $G = f(\mathbb{Q}^2)$ in at most one point.
  - Hence $\mathbb{R}^2 \backslash G$ is separately open set.

Plus open sets

- Example 4:
  - Let $\{B_1, B_2, B_3, ...\}$ be a countable basis for the standard topology on $\mathbb{R}^2$.
  - For each $n \in \mathbb{N}$, inductively choose $(x_n, y_n) \in B_n$ s.t. $\forall m < n, x_n \neq x_m$ and $y_n \neq y_m$.
  - Let $F = \{(x_1, y_1), (x_2, y_2), ...\}$.
  - F Contains a point from every basic open set, so $int\left(\mathbb{R}^2 \backslash F\right) = \phi$
    i.e. $\mathbb{R}^2 \backslash F$ contains no open set.

Claim: $\mathbb{R}^2 \backslash F$ is $+$open.

- Let $(a, b) \in \mathbb{R}^2 \backslash F$. Their can be at most one point in F with $y-$coordinate b.
- We can choose $\varepsilon_1 > 0$ s.t. $|x - a| < \varepsilon_1 \Rightarrow (x, b) \notin F$
- similarly, we can have an ε_2 s.t. $|y - b| < \varepsilon_2 \Rightarrow (a, y) \notin F$.
- Let $\varepsilon = min(\varepsilon_1, \varepsilon_2)$ which implies $+_\varepsilon (a, b) \subseteq \mathbb{R}^2 \backslash F$.

Plus open sets

A G_δ set is a countable intersection of open sets.

- The Maltese Cross A is a G_δ set in euclidean topology.
  - Let $A_n = A \cup B_{1/n}(0,0)$.
  - Each A_n is Euclidean open.
  - $\cap A_n = A$.

Theorem

Let $C \subset \mathbb{R}^2$ is plus open set such that C is euclidean open at all points except those in a set $E \subset C$. If E is G_δ set in euclidean topology, then C is a G_δ set in euclidean topology.

proof.

Since E is a G_δ set, there is a countable collection of Euclidean open sets U_n such that $E = \bigcap U_n$. The set $C_n = C \cup U_n$ is Euclidean open for each n and $C = \cap C_n$.

Plus open sets

Is every Plus open set an Euclidean G_δ set?

Counterexample.

Let $S = \{(x, x) : x \in \mathbb{R}\backslash\mathbb{Q}\}$ and let $G = \mathbb{R}^2\backslash S$ then G is plus-open but not euclidean G_δ set. Because if it were, then $G \cap \{(x, x) : x \in \mathbb{R}\}$ would be a G_δ subset of line $y = x$. But this is impossible as this set is homeomorphic to $\mathbb{Q}$, and $\mathbb{Q}$ is not an G_δ subset of $\mathbb{R}$.

- For any $S = \{(x, x) : x \in E\}$, where E is any subset of $\mathbb{R}$, set $G = \mathbb{R}^2\backslash S$ is Plus-open. $\Rightarrow$ The cardinality of the collection of plus open set is $2^{\mathfrak{c}}$.
- The cardinality of collection of Borel subsets of $\mathbb{R}^2$ is $\mathfrak{c}$. $\Rightarrow$ There must exists plus open sets that are not Borel sets.

Separately Continuous Functions

- Every open set in $\mathbb{R}^2$ is the inverse image of an open set in $\mathbb{R}$ under some continuous function.

  e.g. If $E \subset \mathbb{R}^2$ is open, Let $f(x, y)$ be the distance from (x, y) to set $\mathbb{R}^2 \backslash E$, $\Rightarrow$ $E = f^{-1}((0, \infty))$.

- Is every Plus open set in $\mathbb{R} \otimes \mathbb{R}$ the inverse image of open set in $\mathbb{R}$ under a separately continuous function?

- The cardinality of collection of open sets in $\mathbb{R}$ is $\mathfrak{c}$
- The cardinality of set of separately continuous functions from $\mathbb{R} \otimes \mathbb{R}$ to $\mathbb{R}$ is $\mathfrak{c}$. $\Rightarrow card(\{f^{-1}(E) : E \text{ is open in } R \text{ and } f : \mathbb{R}^2 \to \mathbb{R} \text{ is separately continuous}\}) = \mathfrak{c}$.
- But we have $2^{\mathfrak{c}}$ many plus open sets.

Separation axioms

Definition

Topological space X is a **Hausdorff** if any two distinct points in X are separated by open neighbourhoods.

- The topology on $\mathbb{R} \otimes \mathbb{R}$ is finer than the Euclidean topology, which is a Hausdorff space, so $\mathbb{R} \otimes \mathbb{R}$ is Hausdorff.

Definition

A space X is **Urysohn** if for any two distinct points $x, y \in X$, there exists a continuous function $f : X \to [0, 1]$ s.t. $f(x) = 0$ and $f(y) = 1$.

- A continuous function on $\mathbb{R}^2$ is also continuous on $\mathbb{R} \otimes \mathbb{R}$. Hence the space $\mathbb{R} \otimes \mathbb{R}$ is Urysohn space as $\mathbb{R}^2$ is Urysohn.

Separation axioms

Definition

A space X is **completely Hausdorff** if for any two distinct points $x, y \in X$ there exist disjoint open sets U and V s.t. $x \in U, y \in V$ and $\bar{U} \cap \bar{V} = \phi$.

- if X is Urysohn space then it is completely Hausdorff. therefore $\mathbb{R} \otimes \mathbb{R}$ is completely Hausdorff.

Definition

A space X is called **strongly Hausdorff**, if X is a Hausdorff space and every infinite subset of X contains a sequence $\{x_n\}$ s.t. terms x_n have a pairwise disjoint neighbourhoods in X.

- $\mathbb{R}^2$ is strongly Hausdorff, so $\mathbb{R} \otimes \mathbb{R}$ is also strongly Hausdirff, Since $\mathbb{R} \otimes \mathbb{R}$ is Hausdorff and it has finer topology that $\mathbb{R}^2$.

Separation axioms

Definition

A space X is regular if any close $S \subset X$ and a point $p \in X \setminus S$ are separated by neighbourhoods.

- $\mathbb{R}^2$ is regular, But $\mathbb{R} \otimes \mathbb{R}$ is not regular.
- Construction of proof of the non-regularity of $\mathbb{R} \otimes \mathbb{R}$ is based on the Baire category theorem.
- A topological space X is Baire space iff. every non empty open set U is second category set. i.e. U can not be covered with countable closed sets with empty interior.
- $\mathbb{R} \otimes \mathbb{R}$ is a Baire space.i.e. For any plus open set U, if $U = \cup_{n \in \mathbb{Z}^+} U_n$, then for some n, $\mathfrak{Cl}_+(U_n)$ has non-empty interior. In fact we have slightly stronger result that it has non-empty interior in Euclidean topology.

$\mathfrak{Cl}(X)$: Closure in standard topology.

$\mathfrak{Cl}_+(X)$: Closure in+topology

Non-regularity of $\mathbb{R} \otimes \mathbb{R}$

- If U is non-empty $+$open set, and $U = \cup_{n \in \mathbb{Z}^+} U_n$. Then for some n, $\mathfrak{Cl}_+(U_n)$ has non-empty interior in Euclidean topology.

proof.

Let F be the set as in example-4.
$\Rightarrow \mathbb{R}^2/F$ is $+$open, so F is $+$closed.

Let $(a, b) \notin F$,
Claim:(a, b) and F can not be separated by $+$open sets.
Suppose U and V are disjoint $+$open sets with $(a, b) \in U$ and $F \subset V$,
then $\mathfrak{Cl}_+(U)$ has empty interior in euclidean topology. $\rightarrow\leftarrow$

Neighbourhood base of $\mathbb{Q} \otimes \mathbb{Q}$

Definition

A **second-countable** space is a topological space whose topology has a countable base.

Definition

A space X is said to be **first-countable** if each point has a countable neighbourhood basis. i.e. for each point in X there exist sequence $\{N_i\}_{i=1}^{\infty}$ of neighbourhoods of x s.t. for any neighbourhood N of x, there exist a some N_i contained in a N.

- $\mathbb{R} \otimes \mathbb{R}$ and $\mathbb{Q} \otimes \mathbb{Q}$ are not second countable or even first countable.

A neighbourhood base for a point in $\mathbb{Q} \otimes \mathbb{Q}$ must be uncountable.

- Let $(x, y) \in \mathbb{Q} \otimes \mathbb{Q}$. Assume $\exists$ a countable neighbourhood basis $\{B_n\}$ at (x, y).
- We construct a $+$open set G s.t. $(x, y) \in G$ and for any n, $B_n \not\subset G$ as follows.
- Choose $(x_1, y_1) \in B_1$ s.t. $(x_1, y_1) \neq (x, y)$. Then choose $(x_2, y_2) \in B_2$ s.t. $x_2 \neq x_1, x$ and $y_2 \neq y_1, y$. And so on...
- For any $n \in \mathbb{N}$, if we have selected such $n-1$ points $(x_1, y_1)...(x_{n-1}, y_{n-1})$ from $B_1, B_2, ...B_{n-1}$, since B_n is plus open, we can select $(x_n, y_n) \in B_n$ with $x_n \neq x, x_1..., x_{n-1}$ and $y_n \neq y, ..., y_{n-1}$
- Let $F = \{(x_1, y_1), (x_2, y_2), ...\}$. F contains a point from each B_n. So for any n, $B_n \not\subseteq G = \mathbb{Q}^2 \backslash F$.
- But set G is $+$open. $\rightarrow\leftarrow$

- Note that, there are at most $\mathfrak{c}$ subsets of $\mathbb{Q}^2$. So there are exactly $\mathfrak{c}$ neighbourhoods of a point in $\mathbb{Q} \otimes \mathbb{Q}$.
- Form construction above, there must be uncountable neighbourhood base of a point in $\mathbb{R} \otimes \mathbb{R}$. And indeed it has cardinality $2^{\mathfrak{c}}$.

$\mathbb{Q} \otimes \mathbb{Q}$ is normal

- σset is a separable metric space such that every F_δ set is also a G_δ set
- For any $E \subset \mathbb{Q}$;

$$\boxed{E = \cup_{r \in E}\{r\} = \cap_{r \in \mathbb{Q}\setminus E}\,(\mathbb{Q}\setminus r)}$$

Hence E is both F_σ and G_δ set. $\Rightarrow \mathbb{Q}$ is a σset.

Definition

A topological space X is called zero-dimensional if X is a non-empty T_1-space and has a base consisting of clopen sets.

- $\mathbb{Q}$ is zero dimensional. In fact $\mathbb{Q}$ is strongly zero dimensional.

Definition

A topological space X is called strongly zero-dimensional if X is a non-empty Tychonoff space and every finite functionally open cover $\{U_i\}_{i=1}^{k}$ of the space X has a finite open refinement $\{V_i\}_{i=1}^{m}$ such that $V_i \cap V_j = \phi$ whenever i$\neq j$. Clearly, the refinement $\{V_i\}_{i=1}^{m}$ consists of open-and-closed sets and thus is a functionally open cover of X.

$\mathbb{Q} \otimes \mathbb{Q}$ is normal

- Let fix closed disjoint $H, K \subset \mathbb{Q} \otimes \mathbb{Q}$
- We partition $\mathbb{Q} \otimes \mathbb{Q}$ into clopen U, V with $H \subseteq U$ and $K \subseteq V$.

To do this, inductively construct H_n and K_n s.t.

α $H = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ and $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$

β H_n and K_n closed and disjoint.

γ $\forall x, \exists n$ s.t. $(H_n)_x \cup (K_n)_x = Y$

δ $\forall x, \exists n$ s.t. $\forall y$, $x \in ((H_n)^y)^\circ$ or $x \in ((K_n)_x)^\circ$

- set $U = \cup_n H_n$ and $V = \cup_n K_n$.
- each U_x and U^y is clopen.

Thank You!