Partial Derivatives and Plus Topology

Ashutosh Shinde

26 April 2023

Outline

Introduction

The Plus Topology

Separately Continuous Functions

Partial Derivatives

Constructing examples

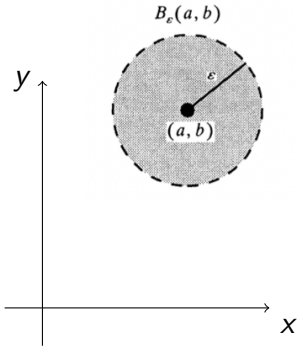
Theorem of Baire

Some Unusual Properties

Standard Topology on $\mathbb{R}^2$:

For $\varepsilon > 0$ and $(a, b) \in \mathbb{R}^2$, the ε-ball centered at (a, b) is set:

$$B_\varepsilon(a, b) = \{(x, y) \in \mathbb{R}^2 | \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon\}$$



- Definitions of continuity and differentiability involve limits in which a point (x, y) can approach (a, b) in any direction.
- Such limits cannot be understood in terms of limits in which the variables x and y approach the limits a and b separately.
- So the partial derivatives cannot be thought of as giving us information about the *'local'* behaviour of the function near a point.

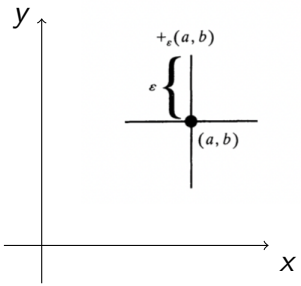
  At least when "*Local*" is defined w.r.t. std. topology.

But if we use different topology?

Plus Topology on $\mathbb{R}^2$

For $\varepsilon > 0$ and $(a, b) \in \mathbb{R}^2$,$+_\varepsilon$ centered at (a, b) is the set:

$$+_\varepsilon(a, b) = \{(x, b) \in \mathbb{R}^2 \mid |x - a| < \varepsilon\} \cup \{(a, y) \in \mathbb{R}^2 \mid |y - b| < \varepsilon\}$$



Definition

$U \subset \mathbb{R}^2$ is plus-open if for every $(a, b) \in U$ there is some $\varepsilon > 0$ s.t. $+_\varepsilon(a, b) \subseteq U$.

plus-open sets form a topology on $\mathbb{R}^2$

Example.:

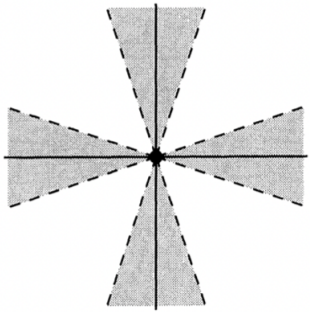
open set : open w.r.t. standard topology.
plus - open : open set w.r.t. plus topology.

Consider a set:

The $+$topology is strictly finer than std. topology.

$$A = \{0,0\} \cup \{(x,y) \mid |y| > 3|x|\} \cup \{(x,y) \mid |y| < \frac{|x|}{3}\}$$

- A is Plus open. but not an open set.
- Also every open set is plus-open.



The set A.

Separately Continuous Functions

Definition

A function f with domain $\mathbb{R}^2$ is called *S*eparately Continuous if for every $b \in \mathbb{R}$, $f(x, b)$ is continuous function of x, and for every $a \in \mathbb{R}$, $f(a, y)$ is continuous function of y.

Theorem

Let Y be topological space, and $f : \mathbb{R}^2 \mapsto Y$, then f is *separately continuous* iff. it is continuous w.r.t. plus topology on $\mathbb{R}^2$. furthermore plus topology is the only topology for which this is true.

proof. ($\Rightarrow$) Let $V \subseteq Y$ be open. we need show that $f^{-1}(V)$ is $+$open in $\mathbb{R}^2$

- Suppose $(a, b) \in f^{-1}(V)$ then $f(a, b) \in V$.
- Since $f(x, b)$ is continuous, $\exists\, \varepsilon_1 > 0$ s.t. if $|x - a| < \varepsilon_1$ then $f(x, b) \in V$.

- Similarly $\exists\ \varepsilon_2 > 0$ s.t. if $|y - b| < \varepsilon_2$ then $f(a, y) \in V$.
- Let $\varepsilon = min(\varepsilon_1, \varepsilon_2)$, then $+_\varepsilon(a, b) \subseteq f^{-1}(V)$

  ($\Leftarrow$) Suppose f is continuous w.r.t. $+$topology.
- Let $(a, b) \in \mathbb{R}^2$ and V be any neighbourhood of $f(a, b)$ in Y.
- So, $(a, b) \in f^{-1}(V)$, and $f^{-1}(V)$ is $+$open.
- So $\exists\ \varepsilon > 0$ s.t. $+_\varepsilon(a, b) \subseteq f^{-1}(V)$
- Which means, if $|x - a| < \varepsilon$ then $f(x, b) \in V$ and if $|y - b| < \varepsilon$ then $f(a, y) \in V$
- So the function $f(x, b)$ is continuous at $x = a$ and $f(x, b)$ is continuous at $y = b$.

Uniqueness:

- Let $\mathcal{T}$ be topology with above property.
- Let Y be $\mathbb{R}^2$ with $+$Topology, and $f : \mathbb{R}^2 \mapsto Y$ be the identity function.
- Under f, Preimage of any $+$open set is $+$open, hence f is continuous w.r.t. $\mathcal{T}_+$
- Therefore f is separately continuous, by the theorem.

- By the hypothesis, f is continuous w.r.t. $\mathcal{T}$.
  i.e. for any $+$open set U, $f^{-1}(U) \in \mathcal{T}$.
- Since f is identity function, $\mathcal{T}$ is at least as fine as $\mathcal{T}_+$.
- Swapping $\mathcal{T}, \mathcal{T}_+$ we get $\mathcal{T}_+$ is at least as fine as $\mathcal{T}$. So $\mathcal{T}$ must be the $+$topology.

Also there are other topologies defined on $\mathbb{R}^2$, along directions other than the coordinate axes, which can be useful for studying continuity and directional derivatives.

Partial Derivatives

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a function,

- If the partial derivatives f_x and f_y are defined everywhere, then f is separately continuous, i.e. f is continuous w.r.t. plus topology.
- First partial derivatives of a function f at a point (a, b) can be computed from the values of f at all points in any ε-plus centered at (a, b).

$\downarrow$ *applying at every point in +open set*

Theorem

Suppose $f, g : \mathbb{R}^2 \mapsto \mathbb{R}$, U is +open set, and $f(a, b) = g(a, b), \forall (a, b) \in U$. Then $f_x(a, b) = g_x(a, b)$ and $f_y(a, b) = g_y(a, b), \forall (a, b) \in U$. Here equality interpreted as either both partial derivatives are undefined or both are defined and they are equal.

$\downarrow$ *applying this theorem repeatedly on the partial derivatives*

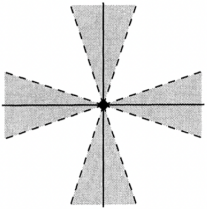
Corollary

Suppose $f, g : \mathbb{R}^2 \mapsto \mathbb{R}$ agree at all points in +open set U, then all partial derivatives (including all mixed partials) agree at all point in U.

Example: Let A be +open set as above

$$f(x,y) = \begin{cases} x^2+y^2 & \text{if } (x,y) \in A \\ -1 & \text{if } (x,y) \notin A \end{cases} \qquad g(x,y) = x^2+y^2$$

- All partial derivatives of f and g agree at $(0,0)$.
- But the *'behaviour'* of these functions is quite different near $(0,0)$.

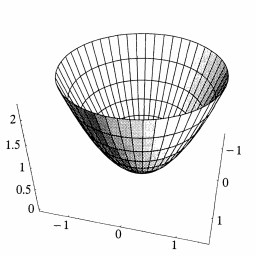


The set A.

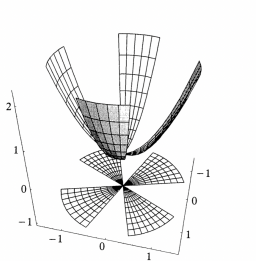
Differentiability and partial derivatives

- g is differentiable at $(0, 0)$, and f is not even continuous there.
- So, we can not say a function is differentiable at a point by inspecting its partial derivatives of any order at that point.
- Test for differentiability: A function is differentiable at a point if the first partial derivatives are defined and **at least one of partial derivative is continuous.**

This example illustrates why the continuity requirement can not be eliminated.



$g(x, y)$



$f(x, y)$

Constructing Examples

Plus topology can be valuable tool in constructing and understanding examples that illustrate the necessity of certain hypotheses.

Theorem: 2^{nd} Derivative Test for Local Extrema

Suppose $f(x,y)$ is differentiable in a neighbourhood of (a,b), $f_x(a,b) = f_y(a,b) = 0$, and f_x and f_y are differentiable at (a,b). Let $\boxed{D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2}$ then:

1. $D > 0, f_{xx}(a,b) > 0 \Rightarrow f$ has local minimum at (a,b).
2. $D > 0, f_{xx}(a,b) < 0 \Rightarrow f$ has local maximum at (a,b).
3. $D < 0 \rightarrow f$ does not have local extremum at has local minimum at (a,b).

Consider function g and f as above:

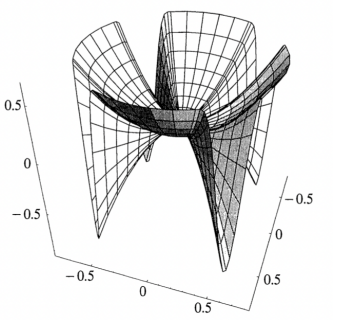
- g has local minimum at $(0,0)$.
- Second Derivative Test "suggests" f has local minimum at $(0,0)$. It is not true!
- Of course, f is not differentiable in neighbourhood of $(0,0)$, so S.D.T. fails.

We can modify f to make it differentiable everywhere, and still the S.D.T. fail.
Let,

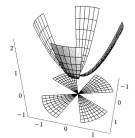
$$z = c(\theta)\, r^2, \quad 0 \le \theta \le 2\pi,\ r^2 = x^2 + y^2$$

We want $c(\theta)$ changes smoothly s.t.

$$c(\theta) : \begin{cases} 1 & \text{for } \theta \text{ close to } n\frac{\pi}{2}. \\ \text{-ve} & \text{for } \theta \text{close to odd multiple of} \frac{\pi}{4} \end{cases}$$



$z = c(\theta) r^2$

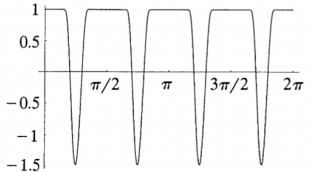


$z = f(x, y)$

Let $C(\theta)$ be periodic function with period $\frac{\pi}{2}$.

Define $c(\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$:

$$c(\theta) = \begin{cases} 1 & \text{if } 0 \leq \theta \leq \pi/8 \\ 1 - \exp\left[6 + \frac{1}{\theta - 3\pi/8} - \frac{1}{\theta - \pi/8}\right] & \text{if } \pi/8 < \theta < 3\pi/8 \\ 1 & \text{if } 3\pi/8 \leq \theta \leq \pi/2 \end{cases}$$

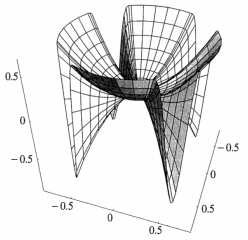


$y = c(\theta)$

Now, let $h(x, y)$ be function s.t. $z = c(\theta) r^2$ be its surface graph.

- h is differentiable everywhere.
- All partial derivatives of h and $g = x^2 + y^2$ agrees at $(0, 0)$.
- But $(0, 0)$ is not local minimum of h. So S.D.T. fails.
- The only hypothesis we haven't checked is differentiability of partial derivatives. And indeed the first partial derivatives are not differentiable at $(0, 0)$.

  one can check $h_x = 2xc(\theta) - yc'(\theta)$ and h_y are not differentiable at $(0, 0)$ as $c(\theta)$ and $c'(\theta)$ are not constant.



$z = c(\theta) r^2$

$+$Open Sets

Are there other exotic $+$ open sets?

- Let $\{B_1, B_2, B_3, ...\}$ be a countable basis for the standard topology on $\mathbb{R}^2$.
- For each $n \in \mathbb{N}$, inductively choose $(x_n, y_n) \in B_n$ s.t. $\forall m < n, x_n \neq x_m$ and $y_n \neq y_m$.
- Let $F = \{(x_1, y_1), (x_2, y_2), ...\}$.
- F Contains a point from every basic open set, so $int\left(\mathbb{R}^2 \backslash F\right) = \phi$
  i.e. $\mathbb{R}^2 \backslash F$ contains no open set.

Claim: $\mathbb{R}^2 \backslash F$ is $+$open.

- Let $(a, b) \in \mathbb{R}^2 \backslash F$. Their can be at most one point in F with $y-$coordinate b.
- We can choose $\varepsilon_1 > 0$ s.t. $|x - a| < \varepsilon_1 \Rightarrow (x, b) \notin F$
- similarly, we can have an ε_2 s.t. $|y - b| < \varepsilon_2 \Rightarrow (a, y) \notin F$.
- Let $\varepsilon = min\left(\varepsilon_1, \varepsilon_2\right)$ which implies $+_\varepsilon (a, b) \subseteq \mathbb{R}^2 \backslash F$.

Define: $c, j : \mathbb{R}^2 \mapsto \mathbb{R}$ as follow:

$$j(x,y) = \begin{cases} 1 & \text{if } (x,y) \in F \\ 0 & \text{if } (x,y) \notin F. \end{cases} \quad \text{and} \quad c(x,y) = 0, \quad \forall (x,y) \in \mathbb{R}^2$$

- j is discontinuous everywhere.
- Since, $j : (x,y) = c(x,y) = 0$, On the $+$open set $\mathbb{R}^2 \backslash F$, all partial derivatives of j are defined and equal to 0 everywhere except countably many points in F.

Can a function have partial derivatives defined everywhere but be discontinuous everywhere?

Baire space:

Some Notations: For $X \in \mathbb{R}^2$,

$$\mathfrak{Cl}(X): \text{ Closure in standard topology.}$$
$$\mathfrak{Cl}_+(X): \text{Closure in} + \text{topology}$$

Note: $\mathfrak{Cl}_+(X) \subseteq \mathfrak{Cl}(X)$

Definition: Baire space

A topological space X is Baire space iff. every non empty open set U is second category set. i.e. U can not be covered with countable closed sets with empty interior.

Theorem

Suppose U is nonempty $+$open set, and $U = \cup_{n \in \mathbb{Z}^+} U_n$. Then for some n, $\mathfrak{Cl}_+(U_n)$ has nonempty interior in standard topology.

Theorem of Baire

Theorem

Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and suppose f_x and f_y are defined at all points in $\mathbb{R}^2$. Then there is dense set of points at which f is differentiable.

proof idea. Claim: For any open set V $(\neq \phi)$, and $\varepsilon > 0$, there is a set W $(\neq \phi)$ s.t :

$$1.\mathfrak{CI}(W) \subseteq V$$
$$2.\ |f_x(\omega_1) - f_x(\omega_2)| < \varepsilon, \forall \omega_1, \omega_2 \in W \quad ...\clubsuit$$

- For any bounded open set V_0, there is point at which f is differentiable.

$|f_x(\omega_1) - f_x(\omega_2)| < \varepsilon, \forall \omega_1, \omega_2 \in W \quad ...\clubsuit$

For open set $V_0, \varepsilon = 1$, by claim We get V_1 s.t.
$\mathfrak{Cl}(V_1) \subseteq V_0$ and V_1 satisfies $\clubsuit$.

$\downarrow$

(set $\varepsilon = 1/2$ apply claim on v_1)

$\downarrow$

We get required V_2.

$\vdots$

For $V_n, \varepsilon = \frac{1}{(n+1)}, \exists V_{n+1}$ s.t.
$\mathfrak{Cl}(V_{n+1}) \subseteq V_n$ and $|f_x(\omega_1) - f_x(\omega_2)| < \frac{1}{(n+1)}$

Let $(a, b) \in \bigcap_{n\in\mathbb{N}} V_n$.

Now for $(x, y) \in V_n$, $|f_x(a, b) - f_x(x, y)| < \frac{1}{n}$
$\Rightarrow f_x$ is continuous at (a, b).

Some Properties:

Theorem

The $+$topology is not regular.

proof. $\mathbb{R}^2/F$ is $+$open, so F is $+$closed.
Let $(a, b) \notin F$,
Claim:(a, b) and F can not be separated by $+$open sets.
Suppose U and V are disjoint $+$open sets with $(a, b) \in U$ and $F \subset V$ then, $\mathfrak{Cl}(U)$ has empty interior. $\rightarrow\leftarrow$

- Plus topology is not second countable or even first countable.
- Also it is difficult to find natural basis for the plus topology.

Thank You!